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CERTAIN PLANE CONTACT PROBLEMS OF ELASTICITY THEORY FOR STRONGLY ANISOTROPIC MEDIA *

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The asymptotic behavior is studied of the solutions of the second boundary value (displacements are given on the boundary) and two contact problems of elasticity theory in a rectangular domain for a curvilinear orthotropic medium with one quite high elasticity coefficient. The sides of the rectangle are parallel to the orthotropy directions. It is shown that for a curvature different everywhere from zero for families of very rigid fibers, we always obtain a medium with inextensible fibers in the limit (the model of the medium with inextensible fibers is introduced in a number of papers, /1/, say). The presence of a large parameter in the generalized Hooke's law results in a singular perturbation of the boundary value problems. Such singularly perturbed problems occur in studying structures from composite materials reinforced by high-modulus fibers /1,2/. It was studied the question of regularity of the degeneration of pre-limiting boundary value problems in limit problems /3/. The uniform asymptotic of the solution in a closed domain contains functions of the angular boundary layer.

1. We take the generalized Hooke's law for the orthotropic material in the form

$$\sigma_{11} = c_{11}l_{11} + c_{12}l_{22}, \quad \sigma_{22} = c_{12}l_{11} + c_{22}l_{22}, \quad \sigma_{12} = 2c_{66}l_{12} \tag{1.1}$$

where l_{11}, l_{12}, l_{22} are deformations in the orthogonal coordinates (x_1, x_2) which we henceforth assume isothermal, to simplify the formulas. The positiveness of the strain potential energy results in the constraints: $c_{ii} > 0,\ i = 1, 2, 6$, and $c_{11}c_{22} - c_{12}^2 > 0$.

Let us introduce the dimensionless stresses and stiffnesses by setting $b_{ij} = c_{ij}c_{66}^{-1}$, $\bar\sigma_{ij} = \sigma_{ij}c_{66}^{-1}$ and let us henceforth conserve the previous notation for the dimensionless stresses. Let us set $b_{11} = \varepsilon^{-2}$ (ε small). Let Q be a rectangular domain on a plane, $Q = \{(x_1, x_2);\ 0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b\}$. Let the curvature of a family of fibers $x_2 \equiv \text{const}$, in $\bar Q$ be strictly different from zero. For the deformations l_{11}, l_{12}, l_{22} we have the relationship

$$l_{11} = \frac{1}{H}\frac{\partial u}{\partial x_1} + \frac{m}{H}v, \quad l_{22} = \frac{1}{H}\frac{\partial v}{\partial x_2} + \frac{k}{H}u$$
$$2l_{12} = \frac{\partial}{\partial x_1}\left(\frac{v}{H}\right) + \frac{\partial}{\partial x_2}\left(\frac{u}{H}\right)$$

where u, v are displacements along the fiber families, respectively

$$x_2 \equiv \text{const} \quad \text{and} \quad x_1 \equiv \text{const}, \quad k = \frac{\partial(\ln H)}{\partial x_1}, \quad m = \frac{\partial(\ln H)}{\partial x_2}$$

where H is the Lamé coefficient.

We set

$$a_1 = b_{12}k_{x_1} - m_{x_2} - bkm - m^2, \quad a_2 = k = bm, \quad a_3 = k_{x_2} + ckm, \quad a_4 = bk_{x_2} - m_{x_1} - ckm$$
$$a_5 = b_{12}m_{x_1} - k_{x_1} - k^2, \quad c = 1 + b_{12}, \quad b = b_{22}$$

The system of equilibrium equations has the form

$$\varepsilon^{-2}\frac{\partial}{\partial x_1}(Hl_{11}) + c\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} + a_1 u + m\frac{\partial v}{\partial x_1} - a_2\frac{\partial v}{\partial x_2} - a_3 v = 0 \tag{1.2}$$
$$c\frac{\partial^2 u}{\partial x_1 \partial x_2} + k(1 + b)\frac{\partial u}{\partial x_2} - m\frac{\partial u}{\partial x_1} + a_4 u +$$

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$$b\frac{\partial^2 v}{\partial x_2^2}+\frac{\partial^2 v}{\partial x_1^2}+a_5 v-H_{x_1}\varepsilon^{-2}l_{11}=0$$

For $x=0$, a are the displacements, u, v satisfy the boundary conditions

$$(0, x_2)=f_1(x_2),\quad v(0, x_2)=f_2(x_2),\quad u(a, x_2)=f_3(x_2) \qquad (1.3)$$
$$v(a, x_2)=f_4(x_2)$$

and for $x=0$, b let one of the following boundary condition combinations be satisfied /4/:

$$u(x_1, 0)=g_1(x_1),\quad v(x_1, 0)=g_2(x_1),\quad u(x_1, b)=g_3(x_1) \qquad (1.4)$$
$$v(x_1, b)=g_4(x_1)$$
$$\tau_{x_1x_2}(x_1, 0)=\tau_{x_1x_2}(x_1, b)=0,\quad v(x_1, 0)=g_2(x_1),\quad v(x_1, b)=g_4(x_1) \qquad (1.5)$$
$$\tau_{x_1x_2}(x_1, 0)=\tau_{x_1x_2}(x_1, b)=0,\ u(x_1, 0)=g_1(x_1),\quad u(x_1, b)=g_2(x_1) \qquad (1.6)$$

We require that the functions f_k, g_k have a sufficiently large quantity of continuous derivatives, and the function $m(x_1x_2)$ be everywhere different from zero in $\bar{Q}$.

We designate the problem of solving the system of equations (1.2) under the boundary conditions (1.3) and (1.4) the problem A_ε, and the problem of solving the system (1.2) under the boundary conditions (1.3) and (1.5), the problem B_ε. The asymptotic behavior of the solutions of problems (1.2), (1.3), (1.6) is constructed analogously to the asymptotic behavior of the problem B_ε and will not be considered here. We consider the question of constructing asymptotics of the solutions of problems A_ε and B_ε for small ε. The smallness of ε means that the domain Q is reinforced by a family of very rigid fibers $x_2\equiv\text{const}$.

2. We construct first the asymptotic of the solution of problem A_ε for small ε. We introduce the function $q=\varepsilon^{-2}l_{11}$ and substitute q into the system of equations (1.2). We seek the approximate solution of the system of equations obtained in the form

$$u(x_1, x_2)=\sum_{n=0}^{N}\varepsilon^n u_n(x_1, x_2),\quad v(x_1, x_2)=\sum_{n=0}^{N}\varepsilon^n v_n(x_1, x_2) \qquad (2.1)$$
$$q(x_1, x_2)=\sum_{n=0}^{N}\varepsilon^n q_n(x_1, x_2)$$

Upon substituting (2.1) into (1.2), we obtain a recurrently coupled system of equations

$$\frac{1}{H}\frac{\partial u_n}{\partial x_1}+\frac{m}{H}v_n=q_{n-2},\quad q_{-2}=q_{-1}=0,\quad n\geqslant 0 \qquad (2.2)$$
$$\frac{\partial}{\partial x_1}(q_nH)+c\frac{\partial^2 v_n}{\partial x_1\partial x_2}+\frac{\partial^2 u_n}{\partial x_2^2}+a_1u_n+m\frac{\partial u_n}{\partial x_1}-a_2\frac{\partial v_n}{\partial x_2}-a_3v_n=0$$
$$c\frac{\partial^2 u_n}{\partial x_1\partial x_2}+k(1+b)\frac{\partial u_n}{\partial x_2}-m\frac{\partial u_n}{\partial x_1}+a_4u_n+b\frac{\partial^2 v_n}{\partial x_2^2}+$$
$$\frac{\partial^2 v_n}{\partial x_1^2}+a_5v_n-H_{x_2}q_n=0$$

Let us recall that the curvature of the families of fibers $x_2\equiv\text{const}$ is proportional to H_{x_2} and is different from zero by assumption. Eliminating q_n from (2.2), we obtain an equation for u_n

$$b\frac{\partial^2 u_n}{\partial x_1^2\partial x_2^2}+\frac{\partial^4 u_n}{\partial x_2^4}+P_n(u_n)=f_n \qquad (2.3)$$

where derivatives of order less than four are in $P_n(u_n)$. Equation (2.3) has a component of the type /5/ with one binary family, of real characteristics $x_2\equiv\text{const}$ in contrast to the system (1.2) which is of elliptic type. The change in the type of system results in the appearance of boundary layer functions along the characteristic part of the boundary in the asymptotic of the solution of the problem A_ε.

We construct a system of equations to determine the boundary layer functions near $x_2=0$. We introduce the stretching coordinate $\eta=x_2/\varepsilon$ into the system of equations (1.2) and we expand the functions $a_k(x_1, \eta\varepsilon)$, $k(x_1, \eta\varepsilon)$, $m(x_1, \eta\varepsilon)$ in Taylor series in powers of ε near $\eta=0$; we set $\rho(x_1)=m(x_1, 0)$. We seek the approximate solution of the system of equations obtained in the form

$$u^{(1)}(x_1, \eta)=\sum_{n=0}^{N}\varepsilon^n u_{n,0}(x_1, \eta),\quad v^{(1)}(x_1, \eta)=\sum_{n=0}^{N}\varepsilon^n v_{n,0}(x_1, \eta)$$

We consequently obtain a recursively related system of equations to determine the boundary layer functions

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u_{n,0}}{\partial x_1}+\rho(x_1)v_{n,0}\right)+\frac{\partial^2 u_{n,0}}{\partial \eta^2}=f_{n,1} \tag{2.4}$$
$$\frac{\partial^2 v_{n,0}}{\partial \eta^2}-\rho(x_1)\left(\frac{\partial u_{n,0}}{\partial x_1}+\rho(x_1)v_{n,0}\right)=f_{n,2}$$

where $f_{0,1}=f_{0,2}=0$. The differential operators $f_{n,1}, f_{n,2}$ are reproduced from the initial system of equations to construct the boundary layer functions. The functions $u_{n,0}, v_{n,0}$ should be noticeably different from zero only near $\eta=0$, hence it is natural to require that they vanish as $\eta\to+\infty$. Consequently, they turn out to be connected by certain relationships.

We first consider the case when $n=0$. We multiply the second equation in the system (2.4) by $\rho^{-1}(x_1)$, differentiate the result with respect to x_1 and combine with the first equation. We obtain a relationship from which it follows that

$$u_{0,0}(x_1,\eta)=-b\frac{\partial}{\partial x_1}[\rho^{-1}(x_1)v_{0,0}(x_1,\eta)]$$

It can be shown by induction that for $n\geqslant 1$

$$u_{n,0}(x_1,\eta)=-b\frac{\partial}{\partial x_1}[\rho^{-1}(x_1)v_{n,0}(x_1,\eta)]+g_{n,0}(v_{n-1,0},\ldots,v_{0,0}) \tag{2.5}$$

where $g_{n,0}$ is a certain uniquely defined function. Substituting (2.5) into the second equation of the system (2.4), we obtain a second order elliptic equation to determine $v_{n,0}(x_1,\eta)$

$$\frac{\partial^2 v_{n,0}}{\partial \eta^2}+\rho(x_1)\frac{\partial^2}{\partial x_1^2}[\rho^{-1}(x_1)v_{n,0}]-\rho^{-2}b^{-1}v_{n,0}=f_{n,3}(v_{n-1,0},\ldots,v_{0,0}) \tag{2.6}$$

where $f_{n,3}=0$ for $n=0$.

The functions $u_{n,1}(x_1,\eta_1), v_{n,1}(x_1,\eta_1)$ of the boundary layer near $x_2=b$ can be constructed in a similar manner by introducing the stretching coordinate $\eta_1=(b-x_2)/\varepsilon$. Here the boundary layer functions satisfy relationships analogous to (2.5).

Therefore, we obtain the asymptotic expansion of the problem in the form

$$u(x_1,x_2)=\sum_{n=0}^{N}\varepsilon^n u_n(x_1,x_2)+\sum_{n=0}^{N}\varepsilon^n\Big\{-b\frac{\partial}{\partial x_1}[\rho^{-1}(x_1)v_{n,0}(x_1,\eta)]+ \tag{2.7}$$
$$g_{n,0}-b\frac{\partial}{\partial x_1}[\rho^{-1}(x_1,b)v_{n,1}(x_1,\eta_1)]+g_{n,1}\Big\}$$
$$v(x_1,x_2)=\sum_{n=0}^{N}\varepsilon^n\left\{-m^{-1}(x_1,x_2)\frac{\partial u_n}{\partial x_1}+m^{-1}Hq_{n-2}+v_{n,0}(x_1,\eta)+v_{n,1}(x_1,\eta_1)\right\}$$

where $v_{n,1}(x_1,\eta_1)$ is the boundary layer function near $x_2=b$, and $\rho^{-1}(x_1,b)=m^{-1}(x_1,b)$. The asymptotic expansions (2.7) permit determination of the boundary conditions for the functions u_n for $x_2=0,b$ and $v_{n,0}, v_{n,1}$ for $\eta=0$ and $\eta_1=0$.

For example, let us note that it is impossible to set

$$u_0(x_1,0)=g_1(x_1),\quad -\rho^{-1}(x_1)\frac{\partial u_0}{\partial x_1}(x_1,0)=g_2(x_1)$$

since it is impossible to give two independent boundary conditions on the characteristic of (2.3); this would result in giving two boundary conditions for the function $v_{n,0}(x_1,\eta)$ at $\eta=0$. But (2.6) is elliptic, and the boundary value problem to determine the boundary layer functions would be incorrect.

To determine the correct boundary conditions, we use the method of elimination. Eliminating u_0 and $v_{0,0}$ successively from (2.7), we obtain that for $x_2=0$ the functions u_0 and $v_{0,0}$ satisfy the boundary conditions

$$v_0(x_1,0)-b\frac{\partial}{\partial x_1}\left[\rho^{-1}(x_1)\frac{\partial u_0}{\partial x_1}(x_1,0)\right]=g_1(x_1)+b\frac{\partial}{\partial x_1}[\rho^{-1}(x_1)g_2(x_1)] \tag{2.8}$$

$$v_{0,0}(x_1,0)-b\frac{\partial}{\partial x_1}\left[\rho^{-1}(x_1)\frac{\partial}{\partial x_1}(\rho^{-1}(x_1)v_{0,0}(x_1,0)\right]=g_2(x_1)+\frac{\partial}{\partial x_1}[\rho^{-1}(x_1)g_1(x_1)]$$

We obtain analogous boundary conditions for $u_0(x_1, x_2)$, $v_{0,1}(x_1, \eta_1)$ for $x_2 = b$. Therefore, none of the boundary conditions of the prelimiting problem A_ε is satisfied in the limit boundary value problem A_0.

Let us note that the representation of the asymptotic of the solution of problem A_ε in the form (2.7) does not permit satisfaction of the boundary conditions (1.4) for $x_1 = 0, a$ since the boundary layer functions can satisfy just one boundary condition for $x_1 = 0, a$ respectively, because of the ellipticity of (2.6). Therefore, to construct the complete asymptotic of the solution of problem A_ε, it is necessary to append boundary layer functions near the angular points of the domain to the functions (2.7).

We construct anagular boundary layer functions near the point $(0, 0)$ (they are determined analogously near the remaining points). We expand the coefficients of the system (1.2) in power series in x_1 and x_2 near the point $(0, 0)$ and we introduce the stretching coordinates $\tau = x_1/\varepsilon$, $\eta = x_2/\varepsilon$. Substituting them into the system (1.2), we obtain

$$\varepsilon^{-2}\left(\varepsilon^{-2}\frac{\partial^2 u}{\partial \tau^2} + \alpha(\tau, \eta) v + \varepsilon^{-1}\frac{\partial v}{\partial \tau}\right) + c\varepsilon^{-2}\frac{\partial^2 v}{\partial \tau \partial \eta} + \varepsilon^{-2}\frac{\partial^2 u}{\partial \eta^2} + \tag{2.9}$$
$$a_1(\tau, \eta) u + m\varepsilon^{-1}\frac{\partial v}{\partial \tau} - a_2(\tau, \eta)\varepsilon^{-1}\frac{\partial v}{\partial \eta} - a_3(\tau, \eta) v = 0$$
$$c\varepsilon^{-2}\frac{\partial^2 u}{\partial \tau \partial \eta} + k(1 + b)\varepsilon^{-1}\frac{\partial u}{\partial \eta} - m\varepsilon^{-1}\frac{\partial u}{\partial \tau} + a_4(\tau, \eta) u +$$
$$b\varepsilon^{-2}\frac{\partial^2 v}{\partial \eta^2} + \varepsilon^{-2}\frac{\partial^2 v}{\partial \tau^2} + a_5(\tau, \eta) v - \varepsilon^2\left(m\varepsilon^{-1}\frac{\partial u}{\partial \tau} + m^2 v\right) = 0$$

Expanding the coefficients of the derivatives in powers of ε, we find the approximate solution of the system (2.9) in the form

$$u^{(2)}(\tau, \eta) = \varepsilon \sum_{n=0}^{N-1} \varepsilon^n p_{n,0}(\tau, \eta), \qquad v^{(2)}(\tau, \eta) = \sum_{n=0}^{N} \varepsilon^n q_{n,0}(\tau, \eta)$$

We then obtain the equations

$$\frac{\partial^2 p_{0,0}}{\partial \tau^2} + m(0, 0)\frac{\partial q_{0,0}}{\partial \tau} = 0 \tag{2.10}$$
$$b\frac{\partial^2 q_{0,0}}{\partial \eta^2} + \frac{\partial^2 q_{0,0}}{\partial \tau^2} - m(0, 0)\frac{\partial p_{0,0}}{\partial \tau} - m^2(0, 0) q_{0,0} = 0$$

for the functions $p_{0,0}$ and $q_{0,0}$, and analogous equations to determine the functions $p_{n,0}$ and $q_{n,0}$ for $n \geqslant 1$.

We require the function $u_0(x_1, x_2)$ to satisfy the boundary conditions (1.3), and the function $v_{0,0}(x_1, \eta)$ the boundary conditions

$$\frac{\partial}{\partial x_1}(\rho^{-1}(x_1) v_{0,0}(x_1, \eta))_{x_1=0,a} = 0 \tag{2.11}$$

Then to determine the functions $p_{0,0}$ and $q_{0,0}$ we obtain the boundary conditions

$$q_{0,0}(0, \eta) = -v_{0,0}(0, \eta), \quad q_{0,0}(\tau, 0) = 0, \quad p_{0,0}(0, \eta) = 0 \tag{2.12}$$

According to conditions (2.12), the bounded solution as $\tau \to +\infty$, for the system of equations (2.10) is uniquely determined.

We obtain boundary conditions analogous to (2.12) for the functions $q_{n,0}$, and $p_{n,0}$ for $n \geqslant 1$. According to boundary conditions (2.8) and (2.11), the function $v_{0,0}(x_1, \eta)$ is defined uniquely, where it damps out exponentially as $\eta \to +\infty$.

Indeed, it can be represented in the form

$$v_{0,0}(x_1, \eta) = \sum_{n=0}^{\infty} e^{-\lambda_n \eta} \psi_n(x_1)$$

where λ_n and $\psi_n(x_1)$ are the eigennumbers and eigenfunctions of the following spectral problem:

$$\rho\frac{d^2}{dx_1^2}[\rho^{-1}(x_1)\psi_n(x_1)] + (\lambda_n^2 - \rho^2 b^{-1})\psi_n(x_1) = 0 \tag{2.13}$$
$$d/dx_1(\rho^{-1}(x_1)\psi_n(x_1))_{x_1=0,a} = 0$$

which is self-adjoint, and as is known, has two series of eigenvalues: for $\lambda_n > 0$ and $\lambda_n < 0$. The requirement of damping as $\eta \to +\infty$ permits keeping only the positive λ_n in the representation mentioned and satisfying the given boundary condition for $\eta = 0$.

3. We briefly consider the problem B_ε, whose asymptotic of the solution is contructed analogously to the asymptotic of the problem A_ε; the distinction is just that for $\eta = 0$ the boundary layer function should be sought in the form

$$u^1(x_1, \eta) = \varepsilon \sum_{n=0}^{N-1} \varepsilon^n u_{n,0}(x_1, \eta), \qquad v^1(x_1, \eta) = \varepsilon \sum_{n=0}^{N-1} \varepsilon^n v_{n,0}(x_1, \eta)$$

However, in contrast to the problem A_ε, the function $v_0(x_1, x_2)$ in the zero-th approximation satisfies the boundary condition (1.5). Therefore, the problem B_ε is regularly degenerate (one of the boundary conditions of the prelimit problem is conserved in the limit).

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